Since several of the initial parameter functions do not remain finite as $b_0 = a_0 \to 0$, a solid (holeless or without inclusion) plate requires special attention. One of several methods of coping with this situation is as follows. Of the initial parameters, $\theta_0 = 0$, w_0 and V_0 have considerable useful physical significance, whereas M_0 has little. The w_0 terms remain finite. V_0 terms can usually be handled by retaining the appropriate initial parameters obtained as explained previously and permitting b_0 to remain small and nonzero until the initial parameters have been evaluated from the boundary conditions. The initial parameter M_0 can be replaced by $M_r(0)$, the moment per unit length, and then setting $C_2 = 0$ in Eq. (4) leads to properly adjusted values of initial parameter functions.

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Body Shape Effects on Skin Friction in Supersonic Flow

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Nomenclature

ν = kinematic viscosity
s = distance along body surface
x = axial distance
r_0 = local body radius
R = transformed body radius
δ = local boundary-layer thickness
δ^* = local boundary-layer displacement thickness
θ = local momentum thickness
H = local body shape parameter defined as δ^*/θ
$\bar{H}$ = dimensionless energy thickness
$H_{p,i}$ = flat-plate shape parameter
δ^{**} = energy thickness
C_f = local skin-friction coefficient
A = constant in momentum thickness equation
u = local velocity
U = transformed inviscid velocity
$U_{s,i}$ = transformed initial velocity
L = shape function
d = dissipation energy
t = turbulence energy
ξ = integrating variable
E = integrating variable
T^* = reference temperature
Re = Reynolds number

Subscripts

0 = stagnation value
e = value at the outer edge of the boundary layer
i = incompressible value
s = initial values
W = properties at the wall
∞ = freestream properties

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THE calculation of the aerodynamic characteristics of aircraft and missiles at high Reynolds numbers requires an accurate knowledge of the contribution of the turbulent skin friction to the total vehicle drag. The skin friction drag C_F of an arbitrary shape exposed to fluid motion requires integrating the local wall shear stress τ_w over the body surface S:

$$C_F = \int_0^s \frac{\tau_w dS}{\frac{1}{2}\rho_\infty U_\infty^2 S} \tag{1}$$

In compressible axisymmetric flow τ_w is related to boundary layer and flow properties through, e.g., the Karman momentum integral equation

$$\frac{\tau_w}{\rho_e} u_e^2 = \frac{C_f}{2} = \frac{d\theta}{ds} + \theta\left[\left(\frac{2+H}{u_e}\right)\left(\frac{du_e}{ds}\right) + \frac{1}{\rho_e}\left(\frac{d\rho_e}{ds}\right) + \frac{1}{r_0}\left(\frac{dr_0}{ds}\right)\right] \tag{2}$$

A direct integration of Eq. (1) requires a knowledge of at least two of the three variables $\theta(s)$, $H(s)$, and $C_f(s)$. Unfortunately, no analytical expression relating pressure gradient to velocity profile, or local skin-friction coefficient to momentum thickness θ and shape parameter H is available. Because of these limitations, most of the methods[1–3] used to calculate turbulent boundary-layer characteristics over bodies with pressure gradients are approximate ones based on the integral forms of the momentum and energy equations in combination with empirically determined relations for skin-friction coefficient, such as the Ludwieg-Tillman relation[4] that includes the effect of shape parameter and boundary-layer velocity profiles. For incompressible flow, Truckenbrodt[5] gives for the momentum thickness θ_i,

$$\theta_i = \left(\frac{1}{U^3R}\right)\left[A\nu_0^{1/6}\int_0^x U^{10/3}R^{7/6}dx\right]^{6/7} \tag{3}$$

After subtracting the momentum equation from the kinetic energy equation and some rearrangements, he obtains also

$$\left(\frac{U_i\theta_i}{\nu_0}\right)^n (\theta_i)\frac{dL_i}{dx} = \left(\frac{U_i\theta_i}{\nu_0}\right)^n\left(\frac{\theta_i}{U_i}\right)\left(\frac{dU_i}{dx}\right) - K(L_i) \tag{4}$$

where

$$K(L_i) = \frac{-\left[2\dfrac{d+t}{U_i^2\rho_0} - H_i\dfrac{\tau_{w,i}}{U_i^2\rho_0}\right]\left[\dfrac{U_i\,\theta_i}{\nu_0}\right]^n}{(H_i - 1)\,\bar{H}_i} \tag{5}$$

and

$$L_i = \int_{\bar{H}_{p,i}}^{\bar{H}_i} \frac{d\bar{H}_i}{(H_i - 1)\bar{H}_i} \tag{6}$$

L_i is arbitrarily set equal to zero for zero pressure gradient flows; whence $H_{p,i}$ is equal to 1.4 for turbulent flows. An examination of existing turbulent flow data[4–8] led to the following approximate expression for $K(L_i)$:

$$K(L_i) = 0.0304\{L_i - [0.0305 \ln(U_i\theta_i/\nu_0)]\} - 0.23 \quad (7)$$

Substituting (7) into (4) results in a first degree linear differential equation that is then solved for L_i:

$$L_i = \left(\frac{\xi_{s,i}}{\xi_i}\right) L_{s,\,i} + \ln\left(\frac{U_i}{U_{s,i}}\right) + \frac{1}{\xi_i}\int_{\xi_{s,i}}^{\xi_i}\left[b - \ln\left(\frac{U_i}{U_{s,i}}\right)\right]d\xi_i \quad (8)$$

where

$$\xi_i = \left[A\nu_0^{1/6}\int_0^x U_i^{10/3}\, R^{7/6}\, dx\right]^4 \quad (9)$$

The Truckenbrodt method of calculating incompressible turbulent boundary-layer characteristics may be extended to compressible flows through the use of some appropriate Stewartson-type transformations. Since the method of Truckenbrodt requires fitting constants to the flat-plate skin-friction data, it is necessary to account for the variation of the turbulent shear stresses with Mach number. Using the reference temperature of Tucker and assuming that the viscosity varies linearly with temperature, the effect of compressibility on the turbulent skin-friction coefficient is

$$C_f/C_{f,\,i} = (T_e/T^*)^{5/7} = [\tfrac{1}{5.0}\,(1 + T_0/T_e)]^{5/7} \quad (10)$$

By curve fitting the theoretical incompressible data of Van Driest for good agreement at the high Reynolds numbers range of 10^7 to 5×10^8 we obtain

$$\bar{C}_F = 0.0296/Re^{1/7} \quad (11)$$

Using the preceding expression for $\bar{C}_F$ in the final Truckenbrodt expression for momentum thickness, corrected for compressibility effects, results in

$$\theta = \frac{1}{\left(\frac{a_e}{a_0}\right)^6 M_e^3\, r}\left[\frac{0.0073\left(\frac{\nu_0}{a_0}\right)^{0.167}}{\left(\frac{a_e}{a_0}\right)^{0.238}} \times \int_0^x M_e^{3.33}\, r^{1.167}\left(\frac{a_e}{a_0}\right)^8 dx\right]^{0.857} \quad (12)$$

Applying a Stewartson transformation to the expression for profile parameter results in

$$L_i = \left(\frac{E_s}{E}\right) L_{i,\,s} + \ln\left(\frac{M_e}{M_{e,\,s}}\right) + \frac{1}{E}\int_{E_s}^{E}\left[b - \ln\left(\frac{M_e}{M_{e,s}}\right)\right]dE \quad (13)$$

where

$$E = B^{4(a_e/a_0)5/21}$$

$$b = 0.0305 \ln\,[(a_e/a_0)^{2/\gamma-1}\,(U\theta/\nu_0)] - 0.23$$

$$B = \frac{0.0073}{(a_e/a_0)^{5/21}}\left(\frac{\nu_0}{a_0}\right)^{1/6}\int_0^x M_e^{10/3} r^{7/6}\left(\frac{a_e}{a_0}\right)^{3\gamma-1/\gamma-1} dx$$

Using this transformed value of L_i, the conventional incompressible profile parameter H_i can then be obtained from the known variations of H_i with L_i. H is calculated by combining the transformed momentum thickness

$$\theta = (a_0/a_e)^{\gamma+1/\gamma-1}\,\theta_i = \left[1 + \left(\frac{\gamma-1}{2}\right)M_e^2\right]^3 \theta_i \quad (14)$$

with the transformed displacement thickness

$$\delta^* = (a_0/a_e)^{\gamma+1/\gamma-1}\,[\delta_i^* + (\gamma - \tfrac{1}{2})M_e^2\,(\delta_i^* + \theta_i)] \quad (15)$$

to give

$$H = (a_0/a_e)^2(H_i + 1) - 1 \quad (16)$$

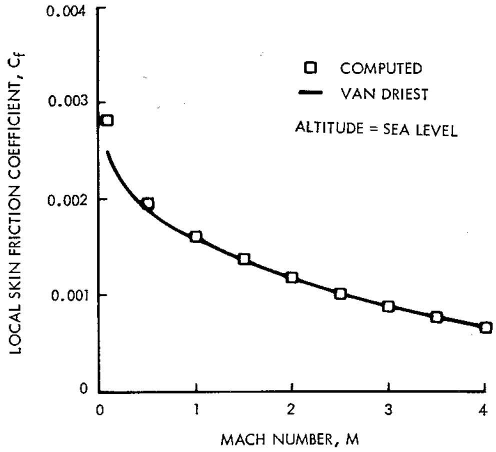

Fig. 1 Comparison of computed and reference data for local skin-friction coefficient on a flat plate.

The calculated shape parameter H and momentum thickness θ are used in the Ludwieg-Tillman skin-friction relation to obtain the local skin-friction distribution over the surface. However, it is first necessary to adapt this relation to compressible flow. Following the procedure of Eckert[1] and Reshotko and Tucker,[2] the transformed relation becomes

$$C_f = 0.246\, e^{-1.561H_i}\,(M_e a_0 \theta_T{}^*/\nu_0)^{-0.268}\,(T_e/T^*)^{0.732}(T_e/T_0)^{0.268} \quad (17)$$

The boundary-layer growth over a series of pointed bodies of revolution with various shapes including cones, ogives, ogive cylinders, "minimum wave drag bodies," and closed bodies of revolution is calculated using the method of the preceding section. To facilitate the calculation of results, the freestream altitude is taken to be sea level, and the maximum body radii are taken to be 1 ft for all configurations. The results from sample calculations for a 20-ft-long flat plate using the present method are compared with data calculated using the method of Van Driest, Fig. 1. Good agree-

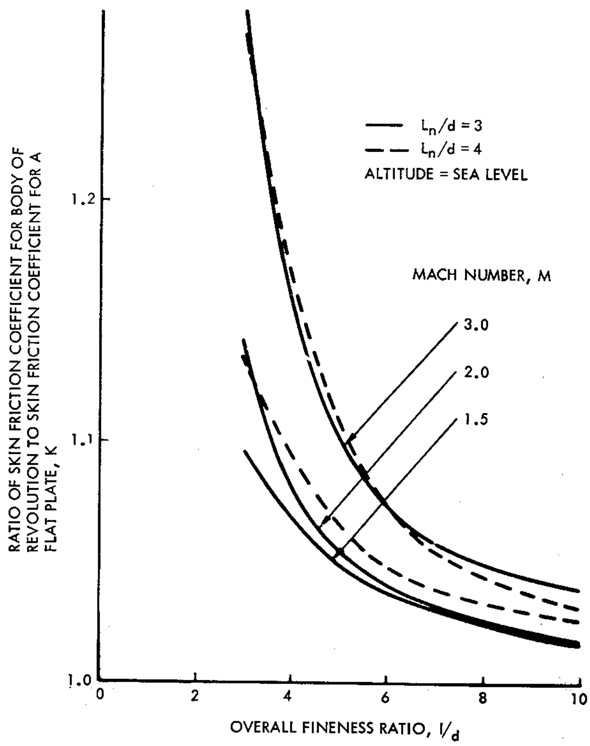

Fig. 2 Effect of fineness ratio on average skin friction for ogive cylinders.

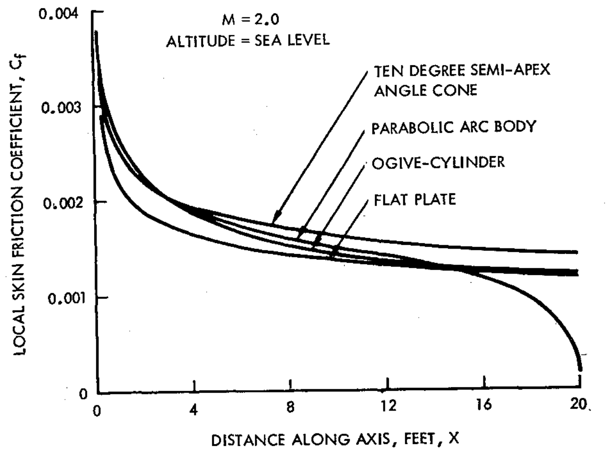

Fig. 3 Body shape effects on local skin-friction distribution.

ment is obtained throughout the supersonic range of $M = 1.0$ to 4.0. Figure 2 shows the effect of varying the over-all fineness ratio for ogive-cylinder combinations at freestream Mach numbers from 1.5 to 3.0.

The effect of ogive-to-cylinder length ratio is small as compared to the effect of the over-all fineness ratio, which affects average skin friction greatly below an over-all fineness ratio of approximately 7. Figure 3 compares the effects of various body shapes on local skin-friction coefficient, for a typical freestream Mach number of 2.0. The ogive cylinder, cone, and parabolic body of revolution show somewhat the same general skin-friction distribution along their forebodies where pressure gradients are favorable ($dp/dx < 0$). The effects of the adverse pressure gradients on the local skin-friction distribution are most pronounced on the parabolic body of revolution which has a rapidly increasing pressure near the aft end. This creates a rapidly falling skin-friction coefficient that approaches the point of separation. The calculation procedure appears to be capable of estimating the separation point, in the case of adverse pressure gradient, although no attempt to verify its accuracy is made.

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Stability of Finite-Difference Equation for the Transient Response of a Flat Plate

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In the course of investigating the transient response of thin shells according to methods outlined by Pian[1] and Leech,[2] the author was led to consider certain finite-difference forms of the equation of motion which governs the transient response of a flat plate. In particular, the problem arose as to what the limits on the ratio of the time mesh to the space mesh might be for an explicit finite-difference formula that would be useful for a numerical calculation of the transient response of such plates.

Fortunately, von Neumann's method of stability analysis as outlined by O'Brien, Hyman, and Kaplan[3] is applicable to this problem, and this analysis (without explanation) is applied herein to the problem at hand.

If one lets D be the customary flexural rigidity of the plate and m be the mass per unit area, the homogeneous plate equation may be written

$$D\nabla^4 w + m\ddot{w} = 0 \tag{1}$$

where w is the displacement normal to the initial plane of the plate. An explicit finite-difference equation, which is equivalent to Eq. (1), may be written as follows,[4] if a "square" mesh $\Delta x = \Delta y$ is used:

$$\begin{aligned}&[D/(\Delta x)^4]\,\{w_{j-2,\,k} - 8w_{j-1,\,k} + 20w_{j,\,k} - 8w_{j+1,\,k} + \\ &w_{j+2,\,k} + 2w_{j-1,\,k+1} - 8w_{j,\,k+1} + 2w_{j+1,\,k+1} + w_{j,\,k+2} + \\ &2w_{j-1,\,k-1} - 8w_{j,\,k-1} + 2w_{j+1,\,k-1} + w_{j,\,k-2}\}_n + \\ &[m/(\Delta t)^2]\{w_{n+1} - 2w_n + w_{n-1}\}_{j,\,k} = 0\end{aligned} \tag{2}$$

In the preceding equation, j and k denote space mesh stations x and y, respectively, and n denotes the instant of time t. Note that Eq. (2) may be used to calculate $w_{n+1,j,\,k}$ when $w_{n,j,\,k}$ and $w_{n-1,\,j,\,k}$ are known.

At least two types of error are associated with such finite-difference equations. The first, truncation error, arises because of the finite distance between points of the finite-difference mesh. This error will not be discussed here since it will be assumed that one is satisfied with the finite-difference approximation. Here then, one is concerned with the second kind of error which is usually considered to be that due to round off associated with using only a finite number of significant figures for any one calculation step; this type of error may grow with increasing time in the stepwise calculation and, if so, would render the numerical solution inaccurate. One wishes to investigate the circumstances under which the error will not grow with time, but instead will die out and thus provide an acceptable solution. This may be regarded as the problem of stability of the finite-difference calculation.

It may be shown[3] that this round off error $\delta(x, y, t)$ must satisfy a similar equation, namely

$$\begin{aligned}&[D/(\Delta x)^4]\{\delta_{j-2,\,k} - 8\delta_{j-1,\,k} + 20\delta_{j,\,k} - 8\delta_{j+1,\,k} + \\ &\delta_{j+2,\,k} + 2\delta_{j-1,\,k+1} - 8\delta_{j,\,k+1} + 2\delta_{j+1,\,k+1} + \\ &\delta_{j,\,k+2} + 2\delta_{j-1,\,k-1} - 8\delta_{j,\,k-1} + 2\delta_{j+1,\,k-1} + \\ &\delta_{j,\,k-2}\}_n + [m/(\Delta t)^2]\{\delta_{n+1} - 2\delta_n + \delta_{n-1}\}_{j,\,k} = 0\end{aligned} \tag{3}$$

Formulas for the difference equivalent to the biharmonic operator, used previously, are given by Crandall.[4] Let it be

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